Lecture 3: Block Theory, Defect Groups, and Decomposition Matrix Structure

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1. Introduction

In this lecture, we explore how the structure of modular representations is refined by decomposing character sets into *blocks*. Each block is associated with a subset of irreducible characters, Brauer characters, and a unique defect group. This gives deep insight into the nature of the decomposition matrix.

2. Blocks of G

Definition 1 (Block)**.** *Let G be a finite group and p a prime dividing $|G|$. A* block *B of G is a primitive idempotent in the center of the group algebra over a suitable modular system. Each block corresponds to a set of irreducible complex characters* $\mathrm{Irr}(B) \subseteq \mathrm{Irr}(G)$ *and a set of Brauer characters* $\mathrm{IBr}(B) \subseteq \mathrm{IBr}(G)$.

The set of blocks is denoted $\mathrm{Bl}(G)$. Every $\chi \in \mathrm{Irr}(G)$ lies in a unique block, and similarly for $\phi \in \mathrm{IBr}(G)$.

3. Block Decomposition Matrix

The decomposition matrix $D = (d_{\chi\phi})$ respects the block decomposition. That is:

$$\text{If } \chi \in \mathrm{Irr}(B) \text{ and } \phi \in \mathrm{IBr}(B'), \text{ with } B \neq B', \text{ then } d_{\chi\phi} = 0.$$

Hence, the global decomposition matrix is block-diagonal with each block corresponding to a separate $B \in \mathrm{Bl}(G)$.

4. Defect Groups

Definition 2 (Defect Group)**.** *Let $B \in \mathrm{Bl}(G)$. A p-subgroup $D \leq G$ is called a* defect group *of B if it controls the size and complexity of representations in the block. The* defect *of a character $\chi \in \mathrm{Irr}(B)$ is defined by:*

$$\mathrm{def}_G(\chi) = n \textit{ such that } \chi(1)_p = \frac{|G|_p}{p^n}.$$

The defect group is then a p-subgroup of order p^n.

Remark 1. *Blocks with trivial defect groups (i.e., defect 0) contain a single irreducible character, which is projective. These are called* blocks of defect zero*.*

5. Example: Symmetric Group S_n

Let $G = S_n$, the symmetric group. The block structure with respect to a prime p is governed by the partition theory of n, and each block corresponds to a p-core partition.

- The decomposition matrices for S_n are known explicitly for small n and exhibit block structure.
- For instance, for S_4 and $p = 2$, there are 2 blocks: one with defect 2 and one with defect 0.

6. Structure of Decomposition Matrix in a Block

Theorem 1. *Let $B \in \mathrm{Bl}(G)$ be a block with defect group D. Then:*

- *The number of irreducible Brauer characters in B is equal to the number of isomorphism classes of simple kG-modules in the block.*
- *The number of projective indecomposables in B equals the number of* $\mathrm{IBr}(B)$*.*
- *The decomposition matrix D_B of the block B has full rank and is square if and only if the block is of full defect.*

Example 1. *Let B be a block with cyclic defect group. Then D_B is known to be triangular under a suitable ordering of rows and columns, and all simple modules are 1-dimensional.*

7. Brauer Correspondence and Local Structure

Blocks are intimately connected to local subgroups.

Theorem 2 (Brauer's First Main Theorem (Sketch))**.** *Each block $B \in \mathrm{Bl}(G)$ with defect group D has a unique* Brauer correspondent *in $N_G(D)$, the normalizer of D, denoted $B_D \in \mathrm{Bl}(N_G(D))$, such that they are "related" via the restriction and induction of characters.*

8. Summary

- The decomposition matrix respects block structure: each block corresponds to a submatrix.
- Defect groups quantify the "size" of blocks and reflect local subgroup structure.
- Blocks of defect zero correspond to projective irreducible characters.
- The study of blocks bridges global representation theory and local subgroup data.

Next Lecture (Lecture 4 Preview)

In Lecture 4, we will:

- Study Brauer's three main theorems in detail,
- Investigate the Cartan matrix and its relation to decomposition matrices,
- Discuss the role of the Loewy series and the structure of projective modules.